Trigonometry - Functions and Graphs Lesson #8:
Transformations of Trigonometric Functions - Part Two

In this lesson we will consider the graphs of the functions whose equations are

$$y = a\sin[b(x-c)] + d \quad \text{and} \quad y = a\cos[b(x-c)] + d$$

and relate them to the graphs of the functions whose equations are $y = \sin x$ and $y = \cos x$.

In the first part of the lesson we concentrate on the effects of the parameters c and d.

Class Ex. #1

a) Describe how the graph of the given function compares to the graph of $y = \sin x$, where x is in degrees.

i) $y = \sin(x - 30°)$ — a horizontal translation 30° right
$x \to x - 30°$

ii) $y = \sin x + 2$ — a vertical translation 2 units up
$y \to y - 2$

iii) $y = \sin(x + 60°) - 1$ — a translation 60° left and 1 unit down
$x \to x + 60°$ $\quad y \to y + 1$

iv) $y - 45 = \sin(x - 45°)$ — a translation 45° right and 45 units up
$x \to x - 45°$ $\quad y \to y - 45$

Note

In trigonometry
- a horizontal translation is called a **horizontal phase shift**, and,
- a vertical translation is called a **vertical displacement**.

Class Ex. #2

Complete the table to describe how the graph of the given function compares to the graph of $y = \sin x$ where x is in radians. Use a graphing calculator if necessary.

Equation	Horizontal Phase Shift	Vertical Displacement
$y = \sin x$	0	0
$y = \sin\left(x - \frac{\pi}{4}\right)$	$\frac{\pi}{4}$ radians right	0
$y = \sin x + 5$	0	5 units up
$y + \pi = \sin\left(x + \frac{3\pi}{2}\right)$	$\frac{3\pi}{2}$ radians left	π units down
$y = \sin(x - c) + d$	c radians right	d units up
$y = a\sin[b(x-c)] + d$	c radians right	d units up

Would you expect similar effects on the graph of $y = a\cos[b(x-c)] + d$? Investigate if necessary.

Yes.

Effects of c and d in $y = a \sin[b(x - c)] + d$ *and* $y = a \cos[b(x - c)] + d$

Changing the parameter "c" on the graphs of
$y = a \sin[b(x - c)] + d$ **and** $y = a \cos[b(x - c)] + d$
results in a horizontal phase shift with the following:

- a horizontal phase shift to the right if $c > 0$
- a horizontal phase shift to the left if $c < 0$

Changing the parameter "d" on the graphs of
$y = a \sin[b(x - c)] + d$ **and** $y = a \cos[b(x - c)] + d$
results in a vertical displacement with the following:

- a vertical displacement up if $d > 0$
- a vertical displacement down if $d < 0$

The vertical displacement is determined from a graph using the formula $d = \dfrac{\text{Max} + \text{Min}}{2}$.

Summary of the Effects of the Parameters a, b, c, and d

For $y = a \sin[b(x - c)] + d$
$y = a \cos[b(x - c)] + d$

amplitude = $|a| = \dfrac{\text{Max} - \text{Min}}{2}$

period = $\dfrac{360°}{|b|}$ (for degree measure)

period = $\dfrac{2\pi}{|b|}$ (for radian measure)

horizontal phase shift = c
- to the right if $c > 0$
- to the left if $c < 0$

vertical displacement = d
- up if $d > 0$
- down if $d < 0$
- $d = \dfrac{\text{Max} + \text{Min}}{2}$

For $y = a \tan[b(x - c)] + d$

amplitude - not applicable

a value represents a vertical stretch of factor $|a|$

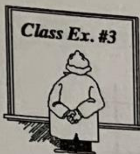
period = $\dfrac{180°}{|b|}$ (for degree measure)

period = $\dfrac{\pi}{|b|}$ (for radian measure)

horizontal phase shift = c
- to the right if $c > 0$
- to the left if $c < 0$

vertical displacement = d
- up if $d > 0$
- down if $d < 0$

Class Ex. #3



Consider equations of the form $y = a \sin[b(x - c)] + d$ and $y = a \cos[b(x - c)] + d$, where $a = 1$, and $b = 1$. Write the equation which represents

a) a cosine function having a horizontal phase shift of 75° right

$y = \cos(x - 75°)$

b) a sine function having a horizontal phase shift of $\frac{3\pi}{5}$ radians left, and a vertical displacement 4 units up

$y = \sin\left(x + \frac{3\pi}{5}\right) + 4$

Class Ex. #4

Find the amplitude, period, horizontal phase shift, and vertical displacement of the graphs of the following functions defined on $x \in R$.

a) $y = 2 \sin 3(x + \pi) - 4$

$a = 2$ amplitude $= 2$ units

$b = 3$ period $= \frac{2\pi}{3}$ radians

$c = -\pi$ horizontal phase shift $= \pi$ radians left

$d = -4$ vertical displacement $= 4$ units down

b) $y = -\frac{2}{3} \cos \frac{1}{4}\left(x - \frac{\pi}{12}\right) + 3$

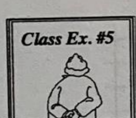
$a = -\frac{2}{3}$ amplitude $= \frac{2}{3}$ unit

$b = \frac{1}{4}$ period $= \frac{2\pi}{1/4} = 8\pi$ radians

$c = \frac{\pi}{12}$ horizontal phase shift $= \frac{\pi}{12}$ radians right

$d = 3$ vertical displacement $= 3$ units up

Class Ex. #5



Find the amplitude, period, horizontal phase shift, and vertical displacement of the graphs of the following functions defined on $x \in R$.

a) $y = 2 \sin(3x + \pi) - 4$

$y = 2 \sin 3\left(x + \frac{\pi}{3}\right) - 4$

$a = 2$ amplitude $= 2$ units

$b = 3$ period $= \frac{2\pi}{3}$ radians

$c = -\frac{\pi}{3}$ horizontal phase shift $= \frac{\pi}{3}$ radians left

$d = -4$ vertical displacement $= 4$ units down

b) $y = -\cos\left(2x - \frac{\pi}{2}\right) + \pi$

$y = -\cos 2\left(x - \frac{\pi}{4}\right) + \pi$

$a = -1$ amplitude $= 1$ unit

$b = 2$ period $= \frac{2\pi}{2} = \pi$ radians

$c = \frac{\pi}{4}$ horizontal phase shift $= \frac{\pi}{4}$ radians right

$d = \pi$ vertical displacement $= \pi$ units up

c) Compare the answer to Class Ex. #4a and Class Ex. #5a.

the only difference is a change in horizontal phase shift

Complete Assignment Questions #1 - #2

Class Ex. #6

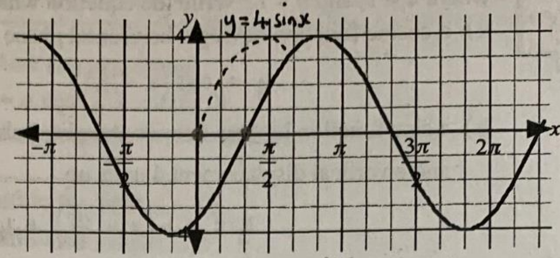
The graphs from a) - d) represent the same trigonometric function.

a) Write the equation of the graph in the form $y = a \sin (x - c)$ if $a > 0$ and there is a minimum possible horizontal phase shift.

amplitude = 4 $\quad a = 4$

h.p.s. = $\frac{\pi}{3}$ right $\quad c = \frac{\pi}{3}$

$y = 4 \sin\left(x - \frac{\pi}{3}\right)$



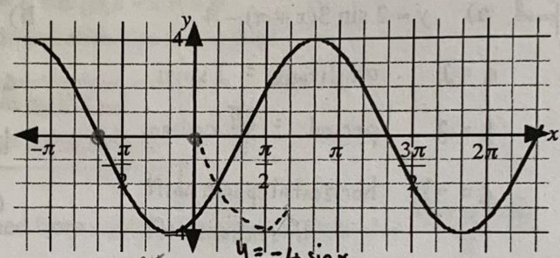
phase shift $\frac{2\pi}{6} = \frac{\pi}{3}$

b) Write the equation of the graph in the form $y = a \sin (x - c)$ if $a < 0$ and there is a minimum possible horizontal phase shift.

amplitude = 4 $\quad a = -4$

h.p.s. = $\frac{2\pi}{3}$ left $\quad c = -\frac{2\pi}{3}$

$y = -4 \sin\left(x + \frac{2\pi}{3}\right)$



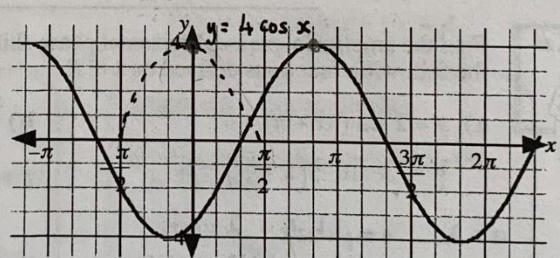
$\frac{4\pi}{6} = \frac{2\pi}{3}$ phase shift

c) Write the equation of the graph in the form $y = a \cos (x - c)$ if $a > 0$ and there is a minimum possible horizontal phase shift.

amplitude = 4 $\quad a = 4$

h.p.s. = $\frac{5\pi}{6}$ right $\quad c = \frac{5\pi}{6}$

$y = 4 \cos\left(x - \frac{5\pi}{6}\right)$



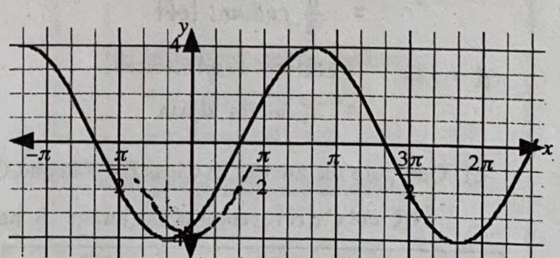
phase shift

d) Write the equation of the graph in the form $y = a \cos (x - c)$ if $a < 0$ and there is a minimum possible horizontal phase shift.

amplitude = 4 $\quad a = -4$

h.p.s. = $\frac{\pi}{6}$ left $\quad c = -\frac{\pi}{6}$

$y = -4 \cos\left(x + \frac{\pi}{6}\right)$

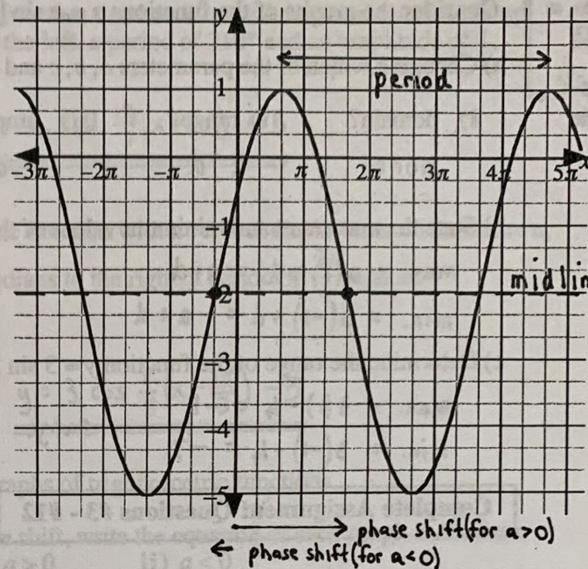


phase shift

Class Ex. #7

Consider the graph shown.

a) If the graph represents a sine function where $a > 0$, complete the table and write the equation represented by the graph.



Amplitude	$\frac{1-(-5)}{2} = 3$
Period	$12\left(\frac{\pi}{3}\right) = 4\pi$
Min. Horizontal Phase Shift	$\frac{\pi}{3}$ rad. left
Vertical Displacement	$\frac{1+(-5)}{2} = -2$

$a = 3$ $\quad b = \frac{2\pi}{4\pi} = \frac{1}{2}$

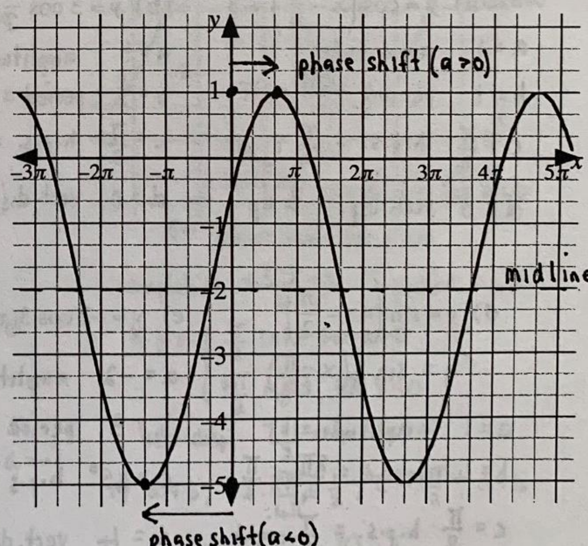
$c = -\frac{\pi}{3}$ $\quad d = -2$

$y = 3\sin\left[\frac{1}{2}\left(x + \frac{\pi}{3}\right)\right] - 2$

b) Write the equation if the graph in a) represents a sine function where $a < 0$.

h.p.s $= \frac{5\pi}{3}$ rad. right $\qquad y = -3\sin\left[\frac{1}{2}\left(x - \frac{5\pi}{3}\right)\right] - 2$

c) If the graph represents a cosine function where $a > 0$, complete the table and write the equation represented by the graph.



Amplitude	3
Period	4π
Min. Horizontal Phase Shift	$\frac{2\pi}{3}$ rad. right
Vertical Displacement	-2

$y = 3\cos\left[\frac{1}{2}\left(x - \frac{2\pi}{3}\right)\right] - 2$

d) Write the equation if the graph in c) represents a cosine function where $a < 0$.

h.p.s. $= \frac{4\pi}{3}$ rad. left $\qquad y = -3\cos\left[\frac{1}{2}\left(x + \frac{4\pi}{3}\right)\right] - 2$

Class Ex. #8

Consider the graphs of the functions $y = a \sin [b(x - c)] + d$ and $y = a \cos [b(x - c)] + d$.

a) Changing which of the parameters a, b, c and d affect the

i) domain?	ii) range?	iii) amplitude?	iv) period?	v) zeros?
none	a, d	a	b	b, c, d (a if d ≠ 0)

b) State the maximum and minimum values of the functions in terms of a, b, c, and d, if $a > 0$.

max. = a(1) + d = a + d

min. = a(−1) + d = −a + d

c) Determine the range of the function $y = 3 \sin 2(x - \pi) - 4$.

max. = 3(1) − 4 = −1

min. = 3(−1) − 4 = −7

range $\{y \mid -7 \le y \le -1, y \in R\}$

Complete Assignment Questions #3 - #12