Trigonometry - Equations and Identities Lesson #2:
Solving Second Degree Trigonometric Equations

In this lesson we will be solving **second degree** equations where the power of the trigonometric function is two (e.g. $\sin^2 x - 3\sin x = 0$).

Trigonometric equations which can be solved by using identities will be covered in lesson 6.

Factoring Trigonometric Expressions

Just as with polynomial expressions, trigonometric expressions can be factored.
The ability to factor trigonometric expressions is a useful skill in two areas:

- solving trigonometric equations (in this lesson)
- proving trigonometric identities (in lesson 6)

In factoring trigonometric expressions we can apply three basic factoring techniques - common factor, difference of two squares, and trinomials of the form $ax^2 + bx + c, a \neq 0$.

Class Ex. #1

Factor the following trigonometric expressions:

a) $8\tan A + 4$ **b)** $\sin^2 x - 3\sin x$ **c)** $4\sin^2 x - 1$

d) $\csc^2 x - 3\csc x - 28$ **e)** $2\cos^2 x + 7\cos x - 4$

Complete Assignment Question #1

Solving a Second Degree Equation Using a Graphical Approach

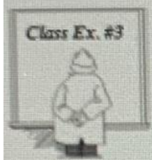
Class Ex. #2

Consider the equation $2\sin^2 x = 1 - \sin x$.

a) Use a **graphical** approach to determine the roots of the equation where $0 \leq x \leq 2\pi$. Give solutions as exact multiples of π.

b) State the general solution to the equation.

Solving a Second Degree Equation Using an Algebraic Approach



Consider the equation $2\sin^2 x = 1 - \sin x$.

a) Use an **algebraic** approach to determine the roots of the equation where $0 \leq x \leq 2\pi$. Give solutions as exact values.

b) State the general solution to the equation.

Use an algebraic procedure to determine the roots of the equation on the given domain and write the general solution.

a) $4\sin^2 A - 1 = 0, 0 \leq A \leq 2\pi$

b) $\tan^2 x + \tan x = 0, \quad 0 \leq x \leq 2\pi$

Class Ex. #5

Determine the zeros of the following functions on the specified domain.

a) $f(x) = \csc^2 x - 3\csc x - 28$,
domain $0° \le x \le 180°$
Answer to the nearest degree

b) $g(\theta) = 2\cos^2\theta + 5\cos\theta - 3$,
domain $-\pi \le \theta \le \pi$

Complete Assignment Questions #2 - #13

Assignment

1. Factor the following trigonometric expressions.

a) $4\sin^2\theta - \cos^2\theta$ **b)** $\cot^2 x - \cot x$ **c)** $\sin^2\theta + 3\sin\theta + 2$

d) $\sec x \sin^2 x - 0.25\sec x$ **e)** $\cot^2\theta - 1$ **f)** $\sec^4\theta - 1$

g) $4\cos^2 A - 4\cos A - 3$ **h)** $2\sin^2 x - 7\sin x + 6$

2. Consider the equation $2\cos^2 x + 3\cos x + 1 = 0$.

   a) Use a **graphical** approach to find the solution to the equation where $0 \le x \le 2\pi$. Give solutions as exact values.

   b) Use an **algebraic** approach to find the solution to the equation where $0 \le x \le 2\pi$. Give solutions as exact values.

   c) State the general solution to the equation.

3. Algebraically find the solutions to the following trigonometric equations. Give solutions as exact values.

   a) $2\sin^2\theta + \sin\theta = 0$ where $0 \le \theta \le 2\pi$

   b) $2\sin^2 x - \sin x = 1$ where $0 \le x \le 2\pi$

Use the following information to answer the next question.

A student is solving the equation $8\cos^2 x + 2\cos x - 3 = 0$ on the interval $0^\circ \le x \le 360^\circ$. The student's work is shown below.

$$8\cos^2 x + 2\cos x - 3 = 0$$

$$(2\cos x - 1)(4\cos x + 3) = 0$$

$$\cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -\frac{3}{4}$$

quadrant 1/4	*quadrant 2/3*
reference angle = 60°	*reference angle = 139°*
$x = 60^\circ$	$x = 180^\circ - 139^\circ$
$x = 360^\circ - 60^\circ$	$x = 180^\circ + 139^\circ$
$x = 60^\circ, 300^\circ$	$x = 41^\circ, 319^\circ$

$$x = 41^\circ, 60^\circ, 300^\circ, 319^\circ$$

4. **a)** Verify algebraically that $x = 60^\circ$ is a solution to the equation.

b) Show that $x = 41^\circ$ does not satisfy the equation.

c) Explain the error in the student's work and provide a correct solution to the problem.

Use the following information to answer the next question.

Christine is determining the roots of the equation $2\sin x\cos x = 3\sin x$ on the domain $0 \le x \le 2\pi$. Her work is shown at the side.	$2\sin x\cos x = 3\sin x$ $\frac{2\sin x\cos x}{\sin x} = \frac{3\sin x}{\sin x}$ $2\cos x = 3$ $\cos x = \frac{3}{2}$ no solution

5. a) Is Christine correct in stating that $\cos x = \frac{3}{2}$ has no solution? Explain.

b) Use a graphical approach to show that the equation $2\sin x\cos x = 3\cos x$ on the domain $0 \le x \le 2\pi$ does have roots. Give solutions as exact values.

c) Identify Christine's error and provide a correct algebraic solution to the problem.

6. A trigonometric function, $f(x)$, has a period of 2π radians.

a) If the roots of the equation $f(x) = 0$ on the domain $0 \le x \le 2\pi$ are $x = a$, $x = b$, and $x = c$, state the general solution to the equation $f(x) = 0$.

b) Use the generalization in 6a) and the solution in 5c) to state the general solution to the equation $2\sin x\cos x = 3\cos x$.

c) The three sets of answers in b) can be simplified to a single set of answers. Write the general solution to the equation $2\sin x\cos x = 3\cos x$ in simplest form.

7. Algebraically find the solutions to the following trigonometric equations.
   Give solutions as exact values.

   a) $\cot^2 A + \cot A = 0$ where $-\pi \le A \le \pi$

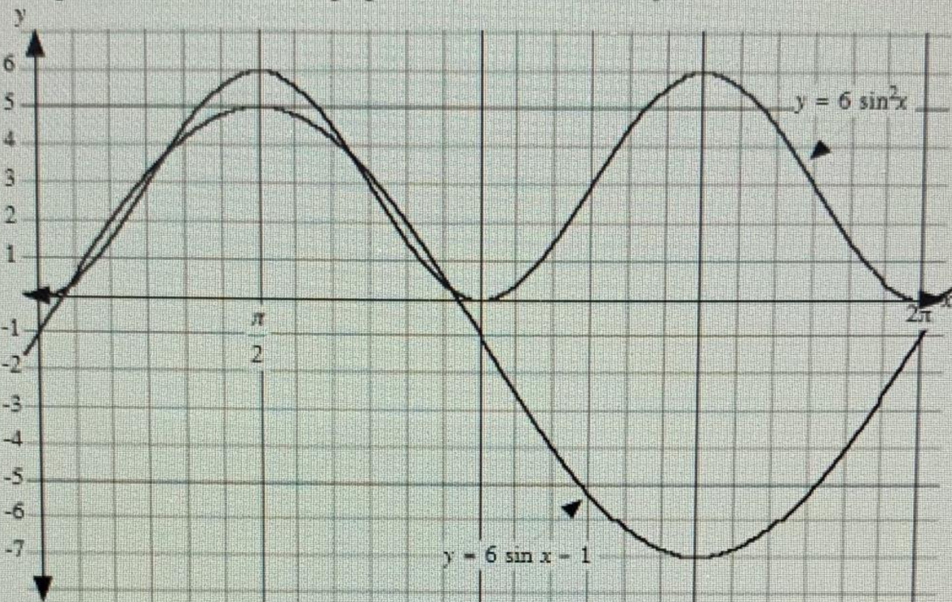
   b) $2\cos^2 x = \sqrt{3}\cos x$ where $-2\pi \le x \le 0$

8. Algebraically find the general solutions to the following trigonometric equations.
   Give solutions as exact values.

   a) $2\csc^2\theta - 2 = 3\csc\theta$

   b) $3\sec\theta = 2 + \sec^2\theta$

9. The diagram below shows the graphs of $y = 6\sin^2 x$ and $y = 6\sin x - 1$ where $0 \le x \le 2\pi$.



a) **Explain how** you could use this diagram to estimate the solution to the equation $6\sin^2 x - 6\sin x + 1 = 0$, where $0 \le x \le 2\pi$.

b) Algebraically determine the solutions to the equation $6\sin^2 x - 6\sin x + 1 = 0$, where $0 \le x \le 2\pi$. Give the solution correct to the nearest hundredth.

c) **Explain how** you could use this diagram to estimate the solution to the equation $6\sin^2 x\,(6\sin x - 1) = 0$, where $0 \le x \le 2\pi$.

d) Use an algebraic approach to find the solutions to the equation $6\sin^2 x\,(6\sin x - 1) = 0$, where $0 \le x \le 2\pi$. Give the solution correct to the nearest hundredth.

10. The following questions cannot be solved using an algebraic approach. Graphically determine the solution(s) on the set of real numbers to the nearest hundredth of a radian.

a) $3\sin^2 x = x$

b) $x^2 + \sin 6x - 1 = 0$

Multiple Choice

11. Which solutions are correct for the equation $12\sin^2 x - 11\sin x + 2 = 0$?

A. $\sin x = 3, 8$

B. $\sin x = \frac{11}{12}, -2$

C. $\sin x = \frac{2}{3}, \frac{1}{4}$

D. $\sin x = -\frac{2}{3}, -\frac{1}{4}$

Numerical Response

12. The number of solutions of the equation $2\cos^2 x + \cos x - 1 = 0$, where $-8\pi \le x \le 8\pi$ is ______ .

(Record your answer in the numerical response box from left to right.)

13. If angle A is acute and $\log_4(\sin^2 A) = -1$, then the value of A, to the nearest tenth of a radian, is ______ .

(Record your answer in the numerical response box from left to right.)

Answer Key

1. a) $(2\sin\theta - \cos\theta)(2\sin\theta + \cos\theta)$ **b)** $\cot x(\cot x - 1)$ **c)** $(\sin\theta + 1)(\sin\theta + 2)$
d) $\sec x(\sin x + 0.5)(\sin x - 0.5)$ **e)** $(\cot\theta - 1)(\cot\theta + 1)$
f) $(\sec\theta - 1)(\sec\theta + 1)(\sec^2\theta + 1)$ **g)** $(2\cos A - 3)(2\cos A + 1)$ **h)** $(\sin x - 2)(2\sin x - 3)$

2. a) $\frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ **b)** $\frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ **c)** $x = \frac{2\pi}{3} + 2n\pi, \pi + 2n\pi, \frac{4\pi}{3} + 2n\pi, n \in I$

3. a) $0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$ **b)** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

4. c) The student has the incorrect reference angle for $\cos x = -\frac{3}{4}$. The correct reference angle is 41°.
The correct solution to the problem is $x = 60°, 139°, 221°, 300°$.

5. a) Christine is correct because the range of the graph of $y = \cos x$ is $-1 \le x \le 1$, and $\frac{3}{2} > 1$.

b) $x = 0, \pi, 2\pi$

c) Christine has divided both sides of the equation by $\sin x$. This is only valid provided $\sin x \ne 0$, but in this question $\sin x = 0$ is a solution to the problem. Her division by $\sin x$ is not a valid step because division by zero is not defined. $x = 0, \pi, 2\pi$.

6. a) $x = a + 2n\pi, b + 2n\pi, c + 2n\pi, n \in I$
b) $x = 2n\pi, \pi + 2n\pi, 2\pi + 2n\pi, n \in I$
c) $x = n\pi, n \in I$

7. a) $-\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ **b)** $-\frac{11\pi}{6}, -\frac{3\pi}{2}, -\frac{\pi}{2}, -\frac{\pi}{6}$

8. a) $x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, n \in I$ **b)** $x = 2n\pi, \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in I$

9. a) Find the x-coordinates of the points of intersection of the two graphs.
b) 0.21, 0.91, 2.23, 2.93
c) Find the x-intercepts of each graph.
d) 0.00, 0.17, 2.97, 3.14, 6.28

10. a) 0.00, 0.35, 2.14 **b)** −1.38, −1.07, −0.63, 0.21, 0.34, 1.04

11. C

12.

2	4		

13.

0	.	5	