Trigonometry - Equations and Identities Lesson #4:
Trigonometric Identities - Part One

Equations and Identities

In mathematics it is important to understand the difference between an equation and an identity.

$2x^2 + 3 = 11$ is an **equation**. It is only true for <u>certain values</u> of the variable x. The solutions to this equation are -2 and 2 which can be verified by substituting these values into the equation.

$(x+1)^2 = x^2 + 2x + 1$ is an **identity.** It is true for <u>all values</u> of the variable x.

Reviewing Identities

Recall the basic trigonometric identities:

Basic Identities

$\sin\theta = \dfrac{y}{r}$ $\quad\cos\theta = \dfrac{x}{r}$ $\quad\tan\theta = \dfrac{y}{x}$ $\quad$ where $x^2 + y^2 = r^2$

We have also met the reciprocal trigonometric identities :

Reciprocal Identities

$\csc x = \dfrac{1}{\sin x}$ $\quad\sec x = \dfrac{1}{\cos x}$ $\quad\cot x = \dfrac{1}{\tan x}$

We can use the Basic and Reciprocal trigonometric identities to prove the Quotient and Pythagorean identities.

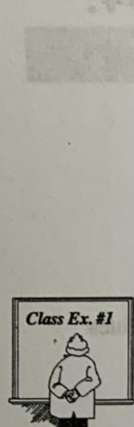
Before doing this we will verify some identities using a particular case.

Verifying Identities for a Particular Case

When verifying an identity we must treat the left side (LS) and the right side (RS) **separately** and work until **both sides** represent the same **value**.

This technique **does not prove** that an identity is true for **all** values of the variable - only for the value of the variable being verified.

Class Ex. #1



Verify the following identities for the value given.

a) $\tan x = \dfrac{\sin x}{\cos x}$ for $x = 60°$

L.S.	R.S.
$\tan 60°$	$\dfrac{\sin 60°}{\cos 60°}$
$= \sqrt{3}$	$= \dfrac{\sqrt{3}/2}{1/2}$
	$= \sqrt{3}$

LS = RS

b) $\tan^2 x + 1 = \sec^2 x$ for $x = \dfrac{\pi}{6}$

L.S.	R.S.
$\left(\tan \frac{\pi}{6}\right)^2 + 1$	$\left(\sec \frac{\pi}{6}\right)^2$
$= \left(\frac{\sqrt{3}}{3}\right)^2 + 1$	$= \dfrac{1}{\left(\cos \frac{\pi}{6}\right)^2} = \dfrac{1}{\left(\sqrt{3}/2\right)^2}$
$= \frac{1}{3} + 1$	$= \dfrac{1}{3/4} = \dfrac{4}{3}$
$= \frac{4}{3}$	

LS = RS

Proving the Quotient Identities and the Pythagorean Identities

We can use identities to derive other identities. When proving an identity we must:
- treat the left side (LS) and the right side (RS) **separately**
- work until **both sides** represent the **same expression**.

Remember:
- Do not make the mistake of assuming the answer by writing the LS = RS at the start of a proof and do not move terms from one side to the other.

Class Ex. #2

Use the basic identities to prove the identity $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$, where $\cos \theta \neq 0$. Explain why there is the restriction $\cos \theta \neq 0$.

L.S.	R.S.
$\tan \theta$	$\dfrac{\sin \theta}{\cos \theta}$
$= \dfrac{y}{x}$	$= \dfrac{y/r}{x/r}$
	$= \dfrac{y}{r} \times \dfrac{r}{x}$
	$= \dfrac{y}{x}$

LS = RS

$\cos \theta \neq 0$ because division by zero is undefined.

Class Ex. #3

Use the basic identities to prove the identity $1 + \tan^2 A = \sec^2 A$.

$$LS = 1 + \left(\frac{y}{x}\right)^2 = 1 + \frac{y^2}{x^2} = \frac{x^2+y^2}{x^2} = \frac{r^2}{x^2}$$

$$RS = \frac{1}{\cos^2 A} = \frac{1}{\left(x/r\right)^2} = \frac{1}{x^2/r^2} = \frac{r^2}{x^2}$$

$LS = RS$

In the same way the basic identities can be used to prove the following:

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

and

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Note

- These identities can be written in several ways and this should be remembered in trying to prove more difficult identities in the next lesson. For example

$$\sin^2 x = 1 - \cos^2 x \qquad \cos^2 x = 1 - \sin^2 x$$

$$\tan^2 x = \sec^2 x - 1 \qquad \cot^2 x = \csc^2 x - 1 \qquad \text{etc.}$$

- We use the basic trigonometric identities in terms of x, y and r to prove **only** the Quotient and Pythagorean Identities.
- You will be asked to verify the remaining Quotient and Pythagorean Identities in the Assignment.
- Before considering more complex identities in the next lesson we need to review some skills in simplification and factoring which will help in the proofs.

Complete Assignment Questions #1 - #5

Using Identities to Simplify Trigonometric Expressions

Class Ex. #4

Express each as a single trigonometric ratio. Use a graphing calculator to verify.

a) $\dfrac{\sin^2 x}{\cos^2 x} + 1 = \dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x} = \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \underline{\underline{\sec^2 x}}$

b) $\sin x + \cot x \cos x = \sin x + \dfrac{\cos x \cdot \cos x}{\sin x} = \dfrac{\sin^2 x}{\sin x} + \dfrac{\cos^2 x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\sin x} = \dfrac{1}{\sin x} = \underline{\underline{\csc x}}$

Class Ex. #5

Express $\dfrac{2 \tan A}{1 + \tan^2 A}$ in terms of $\sin A$ and $\cos A$ and write in simplest form.

$= \dfrac{2\,\dfrac{\sin A}{\cos A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}} = \dfrac{\dfrac{2\sin A}{\cos A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \dfrac{\dfrac{2\sin A}{\cos A}}{\dfrac{1}{\cos^2 A}} = \dfrac{2\sin A}{\cos A} \times \dfrac{\cos^2 A}{1} = \underline{\underline{2\sin A\cos A}}$

Class Ex. #6

Factor the following trigonometric expressions.

a) $3\cos^4\theta - 3\sin^4\theta$

$= 3(\cos^4\theta - \sin^4\theta)$

$= 3(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$

$= 3(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)(1)$

$= \underline{\underline{3(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}}$

b) $\sin^2\theta + \sin^2\theta\cot^2\theta$

$= \sin^2\theta(1 + \cot^2\theta)$

$= \sin^2\theta(\csc^2\theta)$

$= \sin^2\theta\left(\dfrac{1}{\sin^2\theta}\right)$

$= \underline{\underline{1}}$

Complete Assignment Questions #6 - #17