Trigonometry - Equations and Identities Lesson #5:
Trigonometric Identities - Part Two

Exploring an Identity

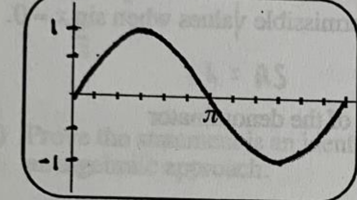
a) Use the quotient identity to show that $\sin x = \tan x \cos x$.

$RS = \tan x \cos x = \frac{\sin x}{\cos x} \cdot \cos x = \sin x = LS$

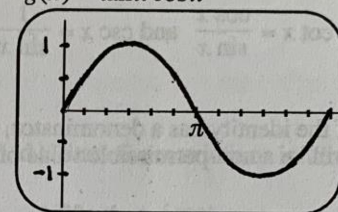
$LS = RS$

b) Is this statement true for all values of x? To investigate, sketch the graphs of $f(x) = \sin x$ and $g(x) = \tan x \cos x$ on the grids below, using a window x:$[0, 2\pi, \frac{\pi}{6}]$ y:$[-1.5, 1.5, 1]$.

$f(x) = \sin x$



$g(x) = \tan x \cos x$



It appears that the graphs are identical and that $f(x) = g(x)$ for all values of x in the interval $0 \le x \le 2\pi$. However, this is not the case as shown below.

c) Evaluate. **i)** $f\left(\frac{\pi}{2}\right)$ and $g\left(\frac{\pi}{2}\right)$ **ii)** $f\left(\frac{3\pi}{2}\right)$ and $g\left(\frac{3\pi}{2}\right)$. What do you notice?

$f\left(\frac{\pi}{2}\right) = 1$ $g\left(\frac{\pi}{2}\right)$ is undefined $f\left(\frac{3\pi}{2}\right) = -1$ $g\left(\frac{3\pi}{2}\right)$ is undefined

The graphs are not identical.

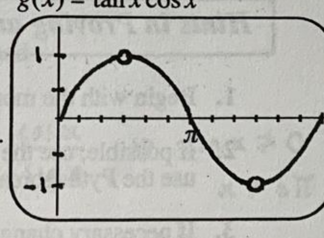
d) The graph of $y = g(x)$ has points of discontinuity at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Explain why, using the results in a).

$\tan \frac{\pi}{2}$ and $\tan \frac{3\pi}{2}$ are undefined

$\tan x = \frac{\sin x}{\cos x}$ and $\cos x = 0$ at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

e) Show the correct graph for $g(x) = \tan x \cos x$ on the grid.

$g(x) = \tan x \cos x$



We say that

$\sin x = \tan x \cos x$ is an identity with non permissible values where $\cos x = 0$.

The restrictions on the identity are $x \neq \frac{\pi}{2} + n\pi, n \in I$.

Note

The "holes" in the graph may not appear on a calculator graph of a trigonometric function. However, using the **Zoom Trig** feature in **degree mode** will usually reveal the points of discontinuity. The graphing calculator may not show the "holes" in radian mode. The axes will need to be turned off if the points of discontinuity lie on one of the axes.

Non-Permissible Values of an Identitiy

The non-permissible values of an identity are determined by finding the non-permissible values for each side of the identity.

When determining the non-permissible values, there are two areas to focus on.

1. All tangent, cotangent, secant, and cosecant functions have non-permissible values.
   - $\tan x = \dfrac{\sin x}{\cos x}$ and $\sec x = \dfrac{1}{\cos x}$ have non-permissible values when $\cos x = 0$.
   - $\cot x = \dfrac{\cos x}{\sin x}$ and $\csc x = \dfrac{1}{\sin x}$ have non-permissible values when $\sin x = 0$.

2. If the identity has a denominator, then any zero of the denominator will be a non-permissible value of the identity.

- The **non-permissible values** are sometimes called the **restrictions** of the identity.
- In many of the examples, the restrictions are the solutions to $\sin x = 0$ and/or $\cos x = 0$.
- Recall $\sin x = 0 \Rightarrow x = n\pi, n \in I$, and $\cos x = 0 \Rightarrow x = \dfrac{\pi}{2} + n\pi, n \in I$, (see page 594).

In this lesson we will verify and prove more complex trigonometric identities using the skills we learned from the previous lesson.
Some useful steps or hints when trying to prove a trigonometric identity are listed below.

Hints in Proving an Identity

1. Begin with the more complex side.

2. If possible, use the reciprocal, quotient, or Pythagorean identities. For example, use the Pythagorean identities when squares of trigonometric functions are involved.

3. If necessary change all trigonometric ratios to sines and/or cosines.
For example, replace $\tan x$ by $\dfrac{\sin x}{\cos x}$, and $\sec x$ by $\dfrac{1}{\cos x}$.

4. Look for factoring as a step in trying to prove an identity.

5. If there is a sum or difference of fractions, combine as a single fraction.

Class Ex. #1

Consider the statement $\frac{1}{\cos x} - \cos x = \sin x \tan x$.

a) Verify the statement is true for $x = \frac{\pi}{3}$.

L.S.	R.S.
$\frac{1}{\cos\frac{\pi}{3}} - \cos\frac{\pi}{3}$	$\sin\frac{\pi}{3}\tan\frac{\pi}{3}$
$= \frac{1}{1/2} - \frac{1}{2}$	$= \frac{\sqrt{3}}{2} \cdot \sqrt{3}$
$= 2 - \frac{1}{2}$	$= \frac{3}{2}$
$= \frac{3}{2}$	

LS = RS

b) Use a graphing calculator to show that the statement is probably an identity.

graph $y_1 = \frac{1}{\cos x} - \cos x$

graph $y_2 = \sin x \tan x$

The graphs should be identical.

c) Prove the statement is an identity using an algebraic approach.

L.S.	R.S.
$\frac{1}{\cos x} - \cos x$	$\sin x \tan x$
$= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}$	$= \sin x \cdot \frac{\sin x}{\cos x}$
$= \frac{1 - \cos^2 x}{\cos x}$	$= \frac{\sin^2 x}{\cos x}$
$= \frac{\sin^2 x}{\cos x}$	

LS = RS

d) State the restrictions in terms of x.

$\tan x = \frac{\sin x}{\cos x}$ so $\cos x \neq 0$

denominator $\cos x \neq 0$

$\cos x \neq 0$

$$x \neq \frac{\pi}{2} + n\pi,\ n \in I$$

Class Ex. #2

Prove the identity $\sin x \cos^2 x + \sin^3 x = \frac{1}{\csc x}$ algebraically and determine the non-permissible values

L.S.	R.S.
$\sin x \cos^2 x + \sin^3 x$	$\frac{1}{\csc x}$
$= \sin x(\cos^2 x + \sin^2 x)$	$= \frac{1}{\frac{1}{\sin x}}$
$= \sin x(1)$	$= 1 \cdot \frac{\sin x}{1}$
$= \sin x$	$= \sin x$

LS = RS

$\csc x = \frac{\cos x}{\sin x}$ so $\sin x \neq 0$

$x \neq n\pi,\ n \in I$

denominator $\csc x \neq 0$

$\frac{\cos x}{\sin x} \neq 0$ so $\cos x \neq 0$

$x \neq \frac{\pi}{2} + n\pi,\ n \in I$

$$x \neq n\pi,\ \frac{\pi}{2} + n\pi,\ n \in I$$

Class Ex. #3

Consider the identity $\dfrac{\sin A + \tan A}{1 + \cos A} = \dfrac{1}{\cot A}$.

a) Prove the identity algebraically.

$$LS = \frac{\sin A + \tan A}{1 + \cos A} = \frac{\sin A + \frac{\sin A}{\cos A}}{1 + \cos A} = \frac{\frac{\sin A \cos A}{\cos A} + \frac{\sin A}{\cos A}}{1 + \cos A} = \frac{\frac{\sin A \cos A + \sin A}{\cos A}}{1 + \cos A}$$

$$= \frac{\frac{\sin A(\cos A + 1)}{\cos A}}{\frac{1 + \cos A}{1}} = \frac{\sin A(\cancel{\cos A + 1})}{\cos A} \cdot \frac{1}{\cancel{1 + \cos A}} = \frac{\sin A}{\cos A} = \tan A = \frac{1}{\cot A} = RS$$

b) What are the non-permissible values of the identity.

$\tan A = \dfrac{\sin A}{\cos A}$ so $\cos A \neq 0$ — $\cos A \neq 0$ $\quad A \neq \frac{\pi}{2} + n\pi,\ n \in I$

$\cot A = \dfrac{\cos A}{\sin A}$ so $\sin A \neq 0$ — $\sin A \neq 0$ $\quad A \neq n\pi,\ n \in I$

$\cos A \neq -1$ $\quad A \neq \pi + 2n\pi,\ n \in I$

denominator $1 + \cos A \neq 0$ so $\cos A \neq -1$

Combine

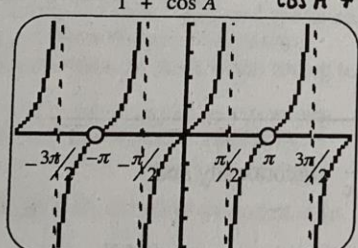
denominator $\cot A \neq 0$ so $\cos A \neq 0$

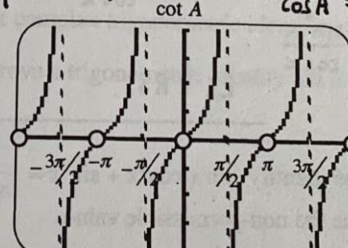
$\underline{A \neq n\pi,\ \frac{\pi}{2} + n\pi,\ n \in I}$

c) Show the non-permissible values on a graph of each side of the identity for the domain $0 \le x \le 2\pi$.

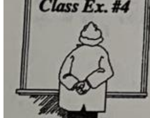
$y = \dfrac{\sin A + \tan A}{1 + \cos A}$ $\quad \cos A \neq 0$, $\cos A \neq -1$

$y = \dfrac{1}{\cot A}$ $\quad \sin A \neq 0$, $\cos A \neq 0$





Class Ex. #4



Prove the identity $\dfrac{\sec^2 x}{\sec^2 x - 1} = \csc^2 x$.

$$LS = \frac{\sec^2 x}{\tan^2 x} = \frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}} = \frac{1}{\cos^2 x} \cdot \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} = \csc^2 x = RS$$

Complete Assignment Questions #1 - #7